Chaos in the \( Z(2) \) gauge model on a generalized Bethe lattice of plaquettes

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Abstract

We investigate the \( Z(2) \) gauge model on a generalized Bethe lattice in the presence of a three-plaquette interaction. We obtain a cascade of phase transitions according to the Feigenbaum scheme leading to chaotic states for some values of parameters of the model. The duality between this gauge model and the three-site Ising spin model on the Husimi tree is shown. The Lyapunov exponents as new order parameters for the characterization of the model in the chaotic region are considered. The line of the continuous phase transition, which corresponds to the points of the first period doubling bifurcation, is also obtained.

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Statistical mechanical models on hierarchical lattices form a large class of exactly soluble models exhibiting a wide variety of phase transitions \([1–3]\). Some types of these lattice models (e.g. Bethe-like) can be used for approximate investigation of macroscopic quantities of the system, while some other types (e.g. diamond-like) approximates the renormalization group transformation (see Ref. \([1]\) for classification). The renormalization group transformation defined on the second types of lattices is a discrete map in the parameter space of the model. The behavior of the renormalization group trajectories and the structure of free energy singularities are closely related. In particular, if renormalization group transformation has some unstable periods or fixed points, then the free energy has nonanalyticities at these points and all these nonanalyticities are the pre-images of the points.

On the other hand, for Bethe-like lattices one can obtain the recursion relation for the partition function. Using the dynamical systems or recursive equations one can obtain an order parameter for the \( Z(N) \) gauge model on a hierarchical lattice. In many cases a recursive sequence converges to a fixed point and one can obtain a qualitatively more correct phase structure than in conventional mean field approximations \([4]\).

It is necessary to mention that for various reasons it...
is interesting to generalize lattice gauge actions by including larger interaction loops. For instance, an enlarged gauge action involving new double plaquette interaction terms was proposed and studied in 3D and 4D [5,6] where qualitative changes in phase diagrams were obtained. The 2D version of one of these lattice gauge models with $Z(2)$ gauge symmetry formulated on the planar rectangular plaquettes was reduced to the usual spin-$1/2$ Ising model on the square lattice and the point of the second-order phase transition was found [7]. Recently, the $Z(3)$ gauge model with double plaquette representation on the flat (triangular and square) [8] and a generalized recursive lattice [9] has been considered and reduced to the spin-1 Blume–Emery–Griffiths model [10].

In this Letter we construct a $Z(2)$ gauge model in the presence of the three-plaquette interaction and show that this model exhibits a cascade of phase transitions according to the Feigenbaum scheme leading to chaotic states for some values of parameters of the model. We point out also the duality between this gauge model on a generalized Bethe lattice of plaquettes and a three-site Ising spin model on the Husimi tree [11].

The generalized Bethe lattice is constructed by successively adding shells. As a zero shell we take the central plaquette and all subsequent shells come out by gluing up two new plaquettes to each link of a previous shell. As a result we get an infinite dimensional lattice on which three plaquettes are gluing up to each link (Fig. 1a). On a lattice the gauge field is described by matrices $U(x,\mu)$ which are assigned to the links of the lattice. The $U(x,\mu)$ are elements of the gauge group itself and $U(x,\mu) = \exp[ia g_0 A_{\mu}(x)]$, where $g_0$ is the bare charge, and $a$ the distance between neighboring sites. In the $Z(2)$ gauge field variables $U_{ij}$ defined on the links take their values among the group of the two roots of unity $U_{ij} \in \{\pm 1\}$. Then the gauge-invariant action in the presence of three-plaquette interaction can be written in the form

$$S = -\beta_3 \sum_{3p} U_{3p} - \beta_1 \sum_{p} U_{p},$$

where

$$U_p = U_{ij} U_{jk} U_{kl} U_{li}$$

is the product of the gauge variables along the plaquette contour and

$$U_{3p} = U_{p1} U_{p2} U_{p3} U_{p4} U_{p5}$$

is the minimal product of the gauge variable along the tree plaquettes (Fig. 1b). The $\beta_1$ and $\beta_3$ are the gauge coupling constants.

The partition function of this model is

$$Z = \sum_{\{U\}} e^{-S},$$

where the sum is over all possible configurations of the gauge field variables $\{U\}$. The expectation value of the central single plaquette $P$ will have the following form,

$$P \equiv \langle U_P \rangle = Z^{-1} \sum_{\{U\}} U_{P} e^{-S}.$$  

The advantage of the generalized Bethe lattice is that for the models formulated on it exact recursion relation can be derived. The partition function separates into four identical branches, when we are cutting apart the zero shell (central plaquette) of the generalized Bethe lattice. Then the partition function for the $n$th generation ($n \to \infty$ corresponds to the thermodynamic limit) can be rewritten

$$Z_n = \sum_{\{U_{Pn}\}} e^{P_i U_{Pn} g_0^4 (U_{Pn})},$$

where the sum is over all possible configurations of the field variables defined on the links of a zero plaquette $\{U_{Pn}\}$ and
\[ g_n(U_{pi}) = \sum_{\{U_{pi+1}^{\prime},U_{pi+1}^{\prime}\prime,\ldots\}} e^{\beta U_{pi+1}^{\prime} + \beta U_{pi+1}^{\prime}\prime + \beta U_{pi+1}^{\prime\prime}} \times g_{n-1}^{3} (U_{pi+1}^{\prime}) g_{n-1}^{3} (U_{pi+1}^{\prime\prime}). \]  

(5)

From Eq. (5) one can obtain
\[
g_n(+) = 16 e^{2\beta_1} g_{n-1}^3 (+) g_{n-1}^3 (-) + 32 e^{2\beta_2} g_{n-1}^3 (-) g_{n-1}^3 (-)
\]
\[
g_n(-) = 16 e^{-2\beta_1} g_{n-1}^3 (+) g_{n-1}^3 (-) + 32 e^{-2\beta_2} g_{n-1}^3 (-) g_{n-1}^3 (-).
\]

Note that \(U_{pi}\) takes the values \(\pm 1\) and coefficients before exponents arise because of the gauge invariance.

After introducing the variable
\[
x_n = \frac{g_n (+)}{g_n (-)},
\]
one can obtain the following recursion relation,
\[
f(x) = \frac{\mu x^2 + 2 \mu x^3 + 3}{\mu x^2 + 2 \mu x^3 + 1},
\]

(7)

where \(z = e^{2\beta_1}, \mu = e^{2\beta_2}\).

Through this \(x_n\) one can express the average value of the central plaquette for the \(n\)th generation,
\[
P_n = \frac{8[ e^{\beta_1} g_n^3 (+) - e^{-\beta_1} g_n^3 (-)]}{8[ e^{\beta_1} g_n^3 (+) + e^{-\beta_1} g_n^3 (-)]}
\]
\[
= \frac{e^{\beta_1} x_n^2 - 1}{e^{\beta_1} x_n^2 + 1}.
\]

(8)

which is the gauge-invariant order parameter in a \(Z(2)\) theory for a stable fixed point in the thermodynamic limit \((n \rightarrow \infty)\).

Note that by using the duality relation, the above recursive relation can be obtained for the magnetization \(m\) of the three-site interacting Ising spin model on the Husimi tree with Hamiltonian \([11]\).

\[-\beta H = J_1 \sum_{\Delta} \sigma_i \sigma_j \sigma_k + h \sum_i \sigma_i,\]

(9)

where \(\sigma_i\) takes values \(\pm 1\), the first sum goes over all triangular faces of the Husimi tree and the second over all sites.

So, we find a relation between the \(Z(2)\) gauge model with the three-plaquette representation of action on a generalized Bethe lattice and the three-site interacting Ising spin model on the Husimi tree. The duality brings the following correspondence,
\[
U_{pi} \leftrightarrow \sigma_i \quad P \leftrightarrow m
\]
\[
U_{pi} U_{pj} U_{pk} \leftrightarrow \sigma_i \sigma_j \sigma_k \quad \beta_1, \beta_2 \leftrightarrow h, J_3.
\]

This duality becomes obvious if one constructs the dual lattice for the generalized Bethe lattice of plaquettes by attaching the nearest centers of plaquettes to each other. Indeed, as a result we get the Husimi tree (Fig. 2).

The plot of \(P\) for different values of \(\beta_1, \beta_3\) \((\beta_3 = -1/kT)\) is presented in Fig. 3 in the thermodynamic limit. For high \(\beta_3\) one has a stable fixed point for \(P\). Decreasing \(\beta_3\) one can obtain a periodic orbit of period \(2\) for \(P\). Further decreasing \(\beta_3\) one can get a periodic orbit of period \(2^n\) or a chaotic attractor which does not have stable periodic orbits. Note that at high and low values of \(\beta_1, P\) completely determines the state of the system for a stable fixed point but for some intermediate values of \(\beta_1\) (corresponding to the chaotic region) we cannot consider \(P\) as an order parameter. The reason for such phase transitions is the different geometrical and dynamical properties of the attracting sets [12] of the map (7) at different values of \(\beta_1\) and \(\beta_3\). The relevant information about the geometrical and dynamical properties of attractors can be found either by using the method of generalized dimensions [13] or investigating the spectrum of Lyapunov exponents. To obtain a good order parameter, one needs a quantity which describes and characterizes qualitative changes at the bifurcation points of band merging, a crisis, or a saddle-node bifurcation of intermittent chaos [14]. Thus, for the characterization of the \(Z(2)\) gauge model on the generalized...
Bethe lattice in the chaotic region one has to consider the generalized dimensions or Lyapunov exponents as order parameters. Recently, we have calculated the spectrum of the Lyapunov exponents of the map given by Eq. (7) in the case of fully developed chaos and shown its nonanalytic behavior [15].

Note that the phase structure of the $Z(2)$ model on the generalized Bethe lattice allows one to take continuum limit at rich sets of $\beta_1, \beta_\beta$. In particular, we solve numerically the following system of equations,

$$f(x) - x = 0, \quad f'(x) = -1,$$

which determines the points of the first period doubling bifurcation, i.e. the points of the continuous phase transitions from the disordered to the two-sublattice ordered phase. The result is presented in Fig. 4.

In summary, we have shown that the $Z(2)$ model on the generalized Bethe lattice exhibits a cascade of phase transitions according to the Feigenbaum scheme in the presence of the three-plaquette interaction. The duality between this gauge model and the three-site Ising spin model on the Husimi tree is shown. We have obtained the line of the continuous phase transitions corresponding to the first period doubling.

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**References**